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NONLINEAR PHENOMENA

Qualitative Models of the Enhanced-Rate Propagation of a Magnetic Field in a Plasma due to the Hall Effect

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Abstract—Two qualitative electron magnetohydrodynamic models are developed of an enhanced-rate (in comparison with ordinary diffusion) propagation of a magnetic field in a plasma due to the Hall effect. The first model is based on a simple hydrodynamic approach, which in particular makes it possible to reproduce some familiar results. The second model provides an exact analytic description of the main global parameters of the enhanced-rate propagation of a magnetic field in an isothermal inhomogeneous plasma: the front velocity of the magnetic field and the effective front width. *© 2000 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The Hall effect in plasmas [1, 2], which is governed by the freezing of a magnetic field in the electron component and, accordingly, by transport of the magnetic field with the electron current velocity, results in an enhanced-rate propagation of the magnetic field, whereas the current flows along the plasma density gradient (to follow the development of this problem, see reviews and papers [3–14]). The electron density can be nonuniform both at the plasma boundary (at the electrode surface) and in the plasma interior. The main mechanism for the rapid penetration of a magnetic field into a plasma is the "braking" of the magnetic field (which is carried by the electric current) at the positive gradient of the electron density. As a result, the magnetic field is "scattered" in the direction orthogonal to the direction of the electron current. This phenomenon has a substantial impact on the dynamics of the field and electron component treated in the electron magnetohydrodynamic (EMHD) theory and on the plasma dynamics treated in a more general MHD approximation. In particular, the rapid penetration of a magnetic field due to the Hall effect plays a key role in the formation of a highly inhomogeneous noncylindrical current sheath in Filippov's plasma focus (PF) discharges. This is confirmed by the good agreement (see [12]) between the experimental data obtained by Orlov *et al.* on the LV-2 device [15] and the results of two-dimensional numerical simulations carried out by Vikhrev and Zabajdullin [11].

In accordance with the hypothesis advanced by Kukushkin and Rantsev-Kartinov [18], the Hall effect plays a particularly important role [16, 17] in the formation of a closed heterogeneous spheromak-like magnetic configuration (SLMC) by the self-magnetic field of Filippov's PF. An analysis of the experimental results of [15] performed by Kukushkin *et al.* [16, 17] provides evidence for the formation of an SLMC with a substantial stored energy. An important feature of the formation of an SLMC in a PF discharge is the possibility of further increasing the stored energy via the compression of the plasma inside the SLMC by the residual magnetic field of the PF. In a hybrid Z–θ pinch that is formed at the major axis of an SLMC, the plasma energy density is substantially (several orders of magnitude) higher than that in experiments on building up force-free spheromak configurations [19] (with the help of an artificially produced poloidal field) and on confining the spheromak plasma in a special chamber of the "flux conserver" type (see, e.g., [20]).

Here, we develop two qualitative models of an enhanced-rate (in comparison with ordinary diffusion) propagation of the magnetic field in a plasma due to the Hall effect. The first model, which develops the "hydrodynamic" approach proposed by Kukushkin [21], makes it possible not only to reproduce some familiar results of EMHD theory, such as the propagation of a magnetic field along the anode surface in a homogeneous plasma (Section 2.1) and the penetration of a magnetic field into a plasma with a step (Section 2.2) density profile and with a density profile increasing monotonically (Section 2.3) in the direction of the current velocity, but also to derive scalings for a plasma with a nonmonotonic density profile in the EMHD model (Section 2.3) and for the initial stage of the plasma displacement from the anode in MHD theory (Section 2.4). The second model provides exact analytic expressions for the main global parameters of the enhanced-rate propagation of a magnetic field in an isothermal inhomogeneous plasma: the front velocity of the magnetic field (Section 3.1) and the effective front thickness (Section 3.2).

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2. "HYDRODYNAMIC" MODEL OF THE ENHANCED-RATE PROPAGATION OF A MAGNETIC FIELD IN A PLASMA

2.1. Enhanced-Rate Propagation of a Magnetic Field along the Anode

In EMHD theory (in which the ion velocity is equal, by definition, to $V_i = 0$), the enhanced-rate propagation of a magnetic field along the anode surface due to the Hall effect can be described by a simple qualitative model proposed by Kukushkin [21], which is based on the qualitative solution to the EMHD equations for an isothermal plasma (Fig. 1),

$$\frac{\partial \mathbf{H}}{\partial t} = -\mathbf{curl}(D_\sigma \operatorname{rot}\mathbf{H}) + \mathbf{curl}[\mathbf{V}_e, \mathbf{H}], \qquad (1)$$

$$\mathbf{V}_e = -\frac{c}{4\pi ne}\mathbf{curl}\mathbf{H}, \qquad (2)$$

with the following initial and boundary conditions: the magnetic field $\mathbf{H} = (0, -H_0, 0)$ at $t = 0$ is nonzero only in the region $X < 0$ and the anode occupies the region $Z < 0$. The EMHD model is based on the following qualitative considerations. A magnetic field diffusely penetrating into the plasma gives rise to a density gradient on the diffusion scale length $\Delta x_{\rm dif}(t)$. By virtue of the freezing of the magnetic field in the electron plasma component, the plasma density gradient–driven electron current with density $j_z = (c/4\pi)(\mathbf{curl}\mathbf{H})_Z = (c/4\pi)(\partial H_y/\partial x)$ carries the magnetic field with a current velocity $V_{eZ} = -j_z/ne$ toward the anode. Near the anode surface, where a high (formally, infinitely high) electron density (and, accordingly, infinitely high electron conductivity) prevents the magnetic field from penetrating into the anode, the electron current changes direction and starts to flow with a velocity $V_{eX}(t) \approx cH_0/4\pi ne\Delta x_{\rm dif}(t)$ along the anode surface, thus carrying the magnetic field with a velocity $\omega_e\tau_{ei}$ times higher than the diffusion velocity. From these considerations, we easily arrive at the following results:

$$\Delta z_{\rm eff} \sim \Delta x_{\rm dif} = \sqrt{2D_\sigma t}, \qquad (3)$$

$$\Delta x_{\rm eff} \sim \sqrt{2D_{\rm eff} t}, \qquad (4)$$

$$D_{\rm eff} = (\omega_e\tau_{ei})^2 D_\sigma, \qquad (5)$$

where

$$D_\sigma = \frac{c^2}{4\pi\sigma}, \qquad (6)$$

τ_{ei} is the electron–ion (e–i) collision frequency, ω_e is the electron gyrofrequency, σ is the plasma conductivity, and D_σ is the magnetic field diffusion coefficient in a plasma.

The qualitative model proposed in [21] reproduces the above formula for $D_{\rm eff}$, which was derived earlier by Gordeev *et al.* [9] from an exact analytic (actually,

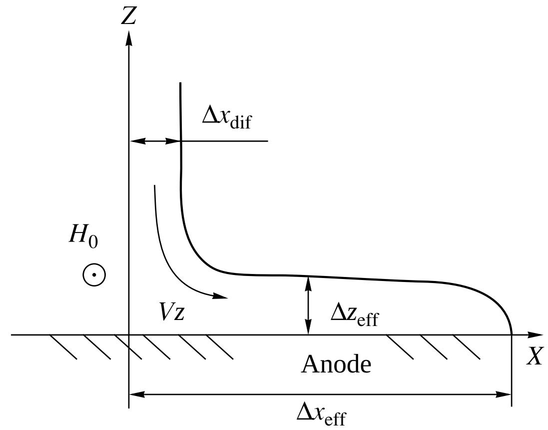

Fig. 1. A graphical illustration of the mechanism for enhanced-rate propagation of a magnetic field along the anode surface.

one-dimensional) analysis of the magnetic field propagation in a narrow layer along the anode (the applicability range of the equations used in [9] was then analyzed in detail by Chukbar [10]). The results obtained in [9], which apply to the propagation of a magnetic field in a plasma in a narrow layer along the anode, were later confirmed by the results of two-dimensional numerical simulations [11], which are valid not only in the anode region but also over the entire plasma volume.

2.2. Propagation of a Magnetic Field along the Boundary between Two Media in the EMHD Model

We consider the propagation of a magnetic field along the boundary between two media with different densities, n_1 and n_2, of free electrons (see the left part of Fig. 2). This is the problem of the magnetic field propagation along the anode surface generalized to the case in which the magnetic field can penetrate into the anode. We consider the simplest case of an isothermal plasma (the method described below can also be used to treat the problem in the case of a nonisothermal plasma).

We consider the magnetic field distribution such that, for $n_2 > n_1$, the electron motion is as shown in the right part of Fig. 2. The motion of the magnetic field front is governed by the current velocities in the first (V_{e1}) and second (V_{e2}) media. If the second medium were an anode with infinite conductivity, then, by the time t, the magnetic field in the first medium would propagate over the distance $\Delta x^{(1)}_{\rm eff} \sim \sqrt{2D^{(1)}_{\rm eff}t}$. However, since the conductivity of the second medium is finite, the electrons in the second medium move in the direction opposite to that of the electron motion in the first medium, causing the magnetic field front to propagate backward through the distance $\Delta x^{(2)}_{\rm eff} \sim \sqrt{2D^{(2)}_{\rm eff}t}$.

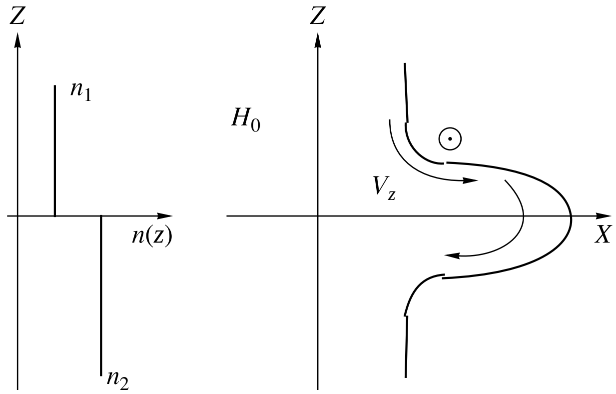

Fig. 2. A graphical illustration of the mechanism for enhanced-rate propagation of a magnetic field along the boundary between two media with different densities of free electrons (on the left). The shape of the field front profile at a certain time is displayed on the right. The arrows show the direction of the electron current.

Consequently, the distance through which the field front propagates along the boundary between the two media with the same temperatures can be estimated as the difference

$$\Delta x_{\text{eff}} \sim \sqrt{2D_{\text{eff}}^{(1)}t} - \sqrt{2D_{\text{eff}}^{(2)}t}. \tag{7}$$

With allowance for (5) and (6), we obtain

$$\Delta x_{\text{eff}}^{(1,2)} \sim \frac{cH_0}{4\pi e}\frac{n_2-n_1}{n_2 n_1}\sqrt{\frac{t}{2D_\sigma}} = \sqrt{2D_{\text{eff}}^{(1,2)}t}, \tag{8}$$

where

$$\sqrt{D_{\text{eff}}^{(1,2)}} \equiv \sqrt{D_{\text{eff}}^{(1)}} - \sqrt{D_{\text{eff}}^{(2)}}, \tag{9}$$

and, as a result, arrive at

$$D_{\text{eff}}^{(1,2)} \sim \left(\frac{\sigma H_0^2}{ec}\frac{n_2-n_1}{n_2 n_1}\right)^2 D_\sigma. \tag{10}$$

In the limit $n_2 \longrightarrow \infty$, formula (10) passes over to (5).

Expression (10) coincides with the relevant expression that was obtained by Vikhrev and Zabajdullin [11,13] using the model that they developed and successfully tested numerically. Note that, in [11, 13], expression (10) was derived for the more general case of media with different conductivities.

2.3. Propagation of a Magnetic Field in a Plasma with a Finite Current-Aligned Density Gradient in the EMHD Model

We consider the dynamic problem of a magnetic field transport by the electric current in a plasma with a small electron-density gradient of fixed sign along the magnetic field front, assuming the plasma temperature to be constant. The results we will obtain from solving this model problem will allow us to derive a qualitative formula describing the motion of the magnetic field front in a plasma with a small density gradient of arbitrary sign, which is regarded as a small density perturbation.

The evolutionary equation for the magnetic field has the form

$$\frac{\partial H}{\partial t} = D_\sigma\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial z^2}\right) - \frac{\partial}{\partial x}(HV_x) - \frac{\partial}{\partial z}(HV_z), \tag{11}$$

where the velocities V_x and V_z depend on the plasma density $n(z)$.

Neglecting diffusion along the magnetic field front, i.e., imposing the condition

$$\frac{\partial^2 H}{\partial x^2} \gg \frac{\partial^2 H}{\partial z^2}, \tag{12}$$

we reduce the initial equation to

$$\frac{\partial H}{\partial t} = D_\sigma\frac{\partial^2 H}{\partial x^2} - \frac{\partial}{\partial x}(HV_x) - \frac{\partial}{\partial z}(HV_z). \tag{13}$$

In the case n = const, equation (13) automatically goes over to a standard one-dimensional equation for the magnetic field diffusion (with the boundary conditions presented in Section 2.1). The diffusive penetration of the field $H^{(0)}$ into a plasma gives rise to a plasma current with the density

$$j_z = -enV_Z^{(0)} = \frac{c}{4\pi}\frac{\partial H^{(0)}}{\partial x} \approx -en\omega_e\tau_{ei}\frac{\partial \Delta x_{\text{dif}}}{\partial t}. \tag{14}$$

where the superscript in the velocity refers to the unperturbed plasma density. Since, in the case at hand, $\partial H/\partial x = 0$, we have $V_x^{(0)} = 0$.

Now, we turn to a plasma in which the density is nonuniform in the direction of the current (i.e., along the Z-axis). We seek the electron current velocity in the form $V_x = V_x^{(0)} + V_x^{(1)}$. Assuming that the density perturbation affects the field dynamics only slightly, we arrive at the relationship

$$\frac{\partial}{\partial x}(HV_X^{(1)}) \sim -\frac{\partial}{\partial z}(HV_Z^{(0)}), \tag{15}$$

which implies that the perturbed current with the velocity $V_x^{(1)}$ is driven exclusively by the plasma density gradient $\partial n/\partial z \neq 0$. Here, we also assume that the transport of the magnetic field by the current causes the “tongue” to extend in the x direction (see Fig. 2) more rapidly than in the case of ordinary diffusion. Then, equation (13) splits into the conventional diffusion equation and equation (15), which describes the magnetic field transport at the electron current velocity. With allowance for (14), we obtain the estimate for $V_x^{(1)}$ and, accordingly, for the velocity of the magnetic field front motion driven by the density gradient $\partial n/\partial z$,

$$\dot{x} \equiv V_X^{(1)} \approx -V_Z^{(0)}\frac{\partial \ln n}{\partial z}\Delta x_{\text{dif}} \approx \omega_e\tau_{ei}D_\sigma\frac{\partial \ln n}{\partial z}, \qquad (16)$$

in which case the field front penetrates the distance

$$\Delta x_{\text{eff}}^{(1)} = \int_0^t V_X^{(1)}(z, t')dt' = \omega_e\tau_{ei}\frac{\partial \ln n}{\partial z}D_\sigma t. \qquad (17)$$

With allowance for the front motion driven by ordinary diffusion, we obtain

$$\Delta x_{\text{eff}} \approx \Delta x_{dif}\left(1 + \varepsilon\omega_e\tau_{ei}\Delta x_{\text{dif}}\frac{\partial \ln n}{\partial z}\right). \qquad (18)$$

The qualitative method used to derive (18) implies that the numerical coefficient ε is of order unity.

Estimate (16) agrees with the results of analytically solving the problem of the penetration of a magnetic field into a plasma whose density increases in the direction of the electron current [6, 7].

To analyze the case in which the electron current substantially changes its direction, i.e., the problem of how the propagation direction of the magnetic field changes in the presence of a localized electron density perturbation (this process may be called the "scattering" of the magnetic field by a localized electron density perturbation), we can apply estimate (16) to a local change in the propagation direction of the magnetic field due to $(\mathbf{j}_e, \partial n/\partial \mathbf{r}) \neq 0$. In this way, it is necessary to transform formula (16) to the frame of reference in which the Z-axis is oriented in the local direction of the vector $\mathbf{j}_e$, so that the magnetic field is scattered through a small angle with respect to this vector. The coordinates of the magnetic field front in the new frame and in the laboratory frame are related by

$$\frac{\partial x}{\partial t} = \frac{\partial \tilde{x}}{\partial t}\cos\alpha - \frac{\partial \tilde{z}}{\partial t}\sin\alpha + \sqrt{\frac{D_\sigma}{2t}}, \qquad (19)$$

$$\sin\alpha = \left|\frac{\partial x}{\partial z}\right|\left[1 + \left(\frac{\partial x}{\partial z}\right)^2\right]^{-1/2}, \quad \cos\alpha = \left[1 + \left(\frac{\partial x}{\partial z}\right)^2\right]^{-1/2},$$

where α is the angle between the tangent to the front at the point at which the new coordinates $(\tilde{Z}, \tilde{X})$ are introduced and the Z-axis (see Fig. 3). With allowance for the relationships

$$\left.\begin{aligned} \frac{\partial \tilde{z}}{\partial t} &\approx \frac{c}{4\pi ne}\frac{H_0}{\Delta\tilde{x}_{\text{dif}}} = \omega_e\tau_{ei}D_\sigma\sqrt{\frac{1}{2D_\sigma t}} \\ \frac{\partial \tilde{x}}{\partial t} &\approx \omega_e\tau_{ei}D_\sigma\frac{\partial \ln n}{\partial z}\cos\alpha \end{aligned}\right\}, \qquad (20)$$

which hold in the new frame, we obtain the following equation, which describes the motion of the magnetic field front $x = x(z, t)$ and is linear in x:

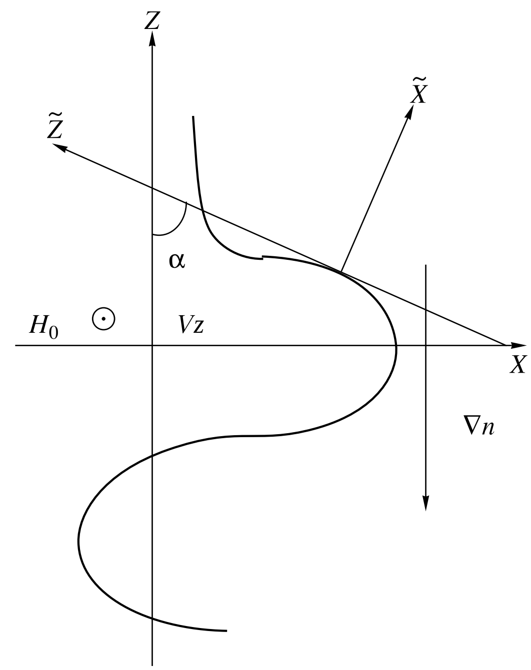

Fig. 3. A schematic representation of the transition from the initial (laboratory) frame to the local frame with the $\tilde{Z}$-axis oriented along the electron current. The arrow indicates the initial direction of the electron density gradient.

$$\frac{\partial x}{\partial t} + \left|\frac{\partial x}{\partial z}\right|\omega_e\tau_{ei}\sqrt{\frac{D_\sigma}{2t}} = \omega_e\tau_{ei}D_\sigma\frac{\partial \ln n}{\partial z} + \sqrt{\frac{D_\sigma}{2t}}, \qquad (21)$$

where the e–i collision frequency $\tau_{ei} = \tau_{ei}(z)$ depends implicitly on z through the plasma density $n(z)$. Equation (21) has the solution

$$\begin{aligned} x &= \omega_e\tau_{ei}\frac{\partial \ln n}{\partial z}D_\sigma t + \sqrt{2D_\sigma t} \\ &+ f\left(\int_{Z_0}^{Z}\frac{dz'}{\omega_e\tau_{ei}(z')} - \sqrt{2D_\sigma t}\right). \end{aligned} \qquad (22)$$

Here, the first term accounts for the contribution of the nonuniform plasma density; the second terms incorporates the contribution of the ordinary diffusion; and the third term describes the transport of the initially perturbed magnetic field front, which is specified through the equation $x(z)|_{t=0} = f\left(\int_{Z_0}^{Z}\frac{dz'}{\omega_e\tau_{ei}(z')}\right)$, by an electron current with the current velocity V_z.

This result generalizes formula (18) (and the corresponding limit of the exact solution obtained by Kingsep *et al.* [7]) to the case of an alternating-sign density gradient $\partial n/\partial z$. Formula (22) describes not only the extension of the tongue in the direction of ordinary diffusion (which results in a faster penetration of the magnetic field into a plasma) but also the appearance of a similar new "tongue," which is stretched in the opposite direction and in which the background magnetic field is reduced (Fig. 3). Formula (22) agrees with the numerical results obtained by Zabajdullin and Vikhrev [14], who simulated the propagation of a magnetic field in a plasma with an alternating-sign density gradient along the current velocity. Also, formula (22) is in qualitative agreement with the results obtained by Chukbar and Yan'kov [8] for the case of a steady-state density profile varying periodically in the direction of the current velocity. Chukbar and Yan'kov [8] showed that, in a steady state, the current flows along snakelike lines, in which case the plasma electric conductivity is $(\omega_e\tau_{ei})^2$ times higher.

The above derivation of the formula describing the evolution of the magnetic field front in a plasma with a small density gradient can be generalized to the case in which the plasma density gradient is arbitrary but the magnetic field diffusion in the direction perpendicular to the field front is still incompletely incorporated. If we take into account the fact that, as the direction of the velocity vector of the electron current at the magnetic field front changes, the absolute value of the velocity vector decreases with time according to the "diffusion" law $(V_z^{(0)}(t))^2 = V_x^2 + V_z^2$, then the velocity component orthogonal to the initial magnetic field front can be estimated as

$$V_x(t) \sim V_z^{(0)}(t)\sin\varphi,$$

$$V_x(t) \approx \Delta x_{\mathrm{dif}} V_z(t)\frac{\partial \ln n}{\partial z}, \quad \tan\varphi = \Delta x_{\mathrm{dif}}\frac{\partial \ln n}{\partial z},$$

where φ is the angle by which the direction of the current velocity changes. This velocity component can be rewritten as

$$V_x(t) \approx V_z^{(0)}\frac{\partial \ln n}{\partial z}\frac{1}{\sqrt{1+\left(\Delta x_{\mathrm{dif}}\frac{\partial \ln n}{\partial z}\right)^2}}, \tag{23}$$

so that the evolution of the magnetic field front is described by the equation

$$x_0 = \omega_e\tau_{ei}\left(\frac{\partial \ln n}{\partial z}\right)^{-1}\left(\sqrt{1+D_\sigma t\left(\frac{\partial \ln n}{\partial z}\right)^2}-1\right). \tag{24}$$

This result agrees qualitatively with the formula

$$x_0 = \frac{1}{2}\omega_e\tau_{ei}D_\sigma t\frac{n_0}{n(z)}\frac{\partial \ln n}{\partial z}\Bigg/\left(1+\frac{\partial \ln n}{\partial z}\sqrt{D_\sigma t}\right) \tag{25}$$

(where n_0 is the unperturbed plasma density), which was deduced by Zabajdullin [22] from the results of two-dimensional numerical simulations. The above formulas reflect the fact that the transition between two limiting regimes of the enhanced-rate propagation of a magnetic field in a plasma due to the Hall effect—specifically, the regime in which the magnetic field propagates as a wave (see [7]) and the regime of diffusive penetration (see [9])—can be described qualitatively by the parameter

$$\mu = \sqrt{D_\sigma t}\frac{\partial \ln n}{\partial z}. \tag{26}$$

Although formulas (24) and (25) are insufficiently accurate for describing the dynamics of the magnetic field front (see Section 3.2 below), they are simple and illustrative and provide a better insight into the transition between different regimes of the enhanced-rate propagation of a magnetic field in a plasma due to the Hall effect.

2.4 Enhanced-Rate Propagation of a Magnetic Field in a Plasma along the Anode Surface with Allowance for Finite Ion Inertia: Plasma Displacement from the Anode

Now, we consider how the enhanced-rate propagation of a magnetic field affects the dynamics of plasma ions (this corresponds to the disruption of the ion's immobility in the sense that ion inertia is taken into account). An understanding of this problem requires solving two-fluid MHD equations (see, e.g., [2, 3, 5, 23, 24]).

For equal constant electron and ion temperatures, $T_e = T_i = T = \mathrm{const}$ [24, 25], and for a magnetic field pressure much higher than the plasma pressure, $H^2/8\pi \gg p$ (the latter condition is valid, in particular, in the initial stages of high-current gas discharges, such as Z-pinch discharges and PF discharges), the Euler equation and continuity equation take the familiar form

$$\left.\begin{aligned}\rho\frac{d\mathbf{u}}{dt} &= -\nabla\frac{H^2}{8\pi}+\frac{1}{4\pi}(\mathbf{H}\nabla)\mathbf{H}\\ \frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\mathbf{u}) &= 0\end{aligned}\right\},$$

where the plasma density ρ and plasma velocity $\mathbf{u}$ are determined by $\rho = m_e n_e + m_i n_i$ and $\mathbf{u} = \dfrac{m_e n_e\mathbf{V}_e + m_i n_i\mathbf{V}_i}{\rho}$, respectively.

For the magnetic field $\mathbf{H} = (0, -H_0, 0)$ and plasma velocity $\mathbf{u} = (u_x, 0, u_z)$ the second term on the right-hand side of the Euler equation vanishes, in which case we have

$$\left.\begin{aligned}\rho\frac{du_x}{dt} &= -\frac{\partial}{\partial x}\frac{H^2}{8\pi}\\ \rho\frac{du_z}{dt} &= -\frac{\partial}{\partial z}\frac{H^2}{8\pi}\end{aligned}\right\}. \tag{27}$$

Using the approach proposed by Kukushkin [21], we can find the scaling describing the plasma displacement from the anode in the above-mentioned initial discharge stage, in which the plasma compression is insignificant and we can set the plasma density to be constant, $\rho = \rho_0 = \text{const}$. Since, near the anode surface, the magnetic field falls off from H_0 to zero along the X-axis (Fig. 1), we have $\partial H/\partial x \approx -H_0/\Delta x_{\text{eff}}$ and, analogously, $\partial H/\partial z \approx -H_0/\Delta z_{\text{dif}}$ (see Section 2.1), which yields

$$\frac{du_x}{dt} \approx \frac{2H_0}{8\pi\rho_0}\frac{2H}{\Delta x_{\text{eff}}} = \frac{H_0^2}{4\pi\rho_0\sqrt{2D_{\text{eff}}t}}.$$

We integrate this equation over t to obtain

$$u_x \approx \frac{H_0^2}{2\pi\rho_0\omega_e\tau_{ei}}\sqrt{\frac{t}{2D_\sigma}}, \tag{28}$$

$$u_z \approx \frac{H_0^2}{2\pi\rho_0}\sqrt{\frac{t}{2D_\sigma}}. \tag{29}$$

A comparison between (28) and (29) gives

$$u_z \simeq (\omega_e\tau_{ei})u_x. \tag{30}$$

Consequently, in the initial stage, the velocity u_z at which the plasma is displaced from the anode is $\omega_e\tau_{ei}$ times higher than the velocity at which the plasma is "dragged" along the anode surface. Thus, we can conclude that the efficiency with which the plasma is displaced from the anode surface because of the magnetic field penetration into the anode region is much higher than the efficiency with which the plasma is dragged by the field.

The perturbed plasma density near the anode can be estimated in an analogous manner from the continuity equation in which the plasma density is sought in the form $\rho(t) \approx \rho_0(1 + \delta(t))$ with $\delta(t) \ll 1$. In the initial stage of the plasma displacement from the anode surface ($t \ll 1/\omega_e\omega_i\tau_{ei}$), the plasma density evolves according to the law

$$\rho(t) \approx \rho_0(1 - \omega_e\tau_{ei}\omega_i t). \tag{31}$$

3. GLOBAL PARAMETERS OF THE MAGNETIC FIELD PROPAGATION IN A PLASMA

We consider a two-dimensional problem of the propagation of a magnetic field in an isothermal plasma in the EMHD model. We assume that at the initial time $t = 0$, the magnetic field $\mathbf{H} = (0, -H_0, 0)$ occupies the region $x < 0$ in a plasma with density $n = n(z)$. With allowance for (2), equation (11), which describes the magnetic field dynamics, becomes

$$\frac{\partial H}{\partial t} = D_\sigma\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial z^2}\right) - \frac{cH}{4\pi e}\frac{\partial H}{\partial x}\frac{\partial}{\partial z}\left(\frac{1}{n}\right). \tag{32}$$

When the plasma density gradient along the magnetic field front in a plasma is small, we can neglect magnetic field diffusion in the Z direction, in which case, with allowance for the inequality $\frac{\partial^2 H}{\partial x^2} \gg \frac{\partial^2 H}{\partial z^2}$, equation (32) can be simplified to

$$\frac{\partial H}{\partial t} = D_\sigma\frac{\partial^2 H}{\partial x^2} - \frac{cH}{4\pi e}\frac{\partial H}{\partial x}\frac{\partial}{\partial z}\left(\frac{1}{n}\right). \tag{33}$$

This equation has an exact analytic solution [7] describing the propagation of the magnetic field in a plasma in the form of a wave,

$$H = \frac{H_0}{e^{\frac{x-ut}{\lambda}} + 1}, \tag{34}$$

where the front width is characterized by the parameter

$$\lambda = \left[\frac{\sigma H_0}{2ec}\frac{\partial}{\partial z}\left(\frac{1}{n}\right)\right]^{-1} \tag{35}$$

and the front velocity is

$$u = -\frac{cH_0}{8\pi e}\frac{\partial}{\partial z}\left(\frac{1}{n}\right). \tag{36}$$

In a plasma with an arbitrary density gradient, equation (32) cannot be solved analytically, meaning only numerical results have been obtained [11, 13, 14, 22]. However, it turns out that such global parameters of the magnetic field dynamics as the depth of the penetration of a magnetic field into a plasma and the effective width of the magnetic field front can be described analytically.

3.1. *Velocity of the Enhanced-Rate Penetration of a Magnetic Field into a Plasma and the Penetration Depth*

We start by imposing the conditions (Fig. 4)

$$H(z, x, t)|_{x\to-\infty} = H_0, \quad H(z, x, t)|_{x\to+\infty} = 0, \quad \left.\frac{\partial H}{\partial x}\right|_{x\to+\infty} = 0 \tag{37}$$

and introducing the effective penetration depth x_0,

$$\int_{-\infty}^{+\infty}(H(x, z, t) - H_0 h(x_0 - x)) = 0, \tag{38}$$

where $h(x)$ is a step function of unit height.

As will be shown below, definition (38) correlates reasonably well with the solutions to equation (33) and with the results obtained for the case of a steep density gradient. Definition (38) and conditions (37) will enable us to derive a closed differential equation for the penetration depth of the magnetic field into a plasma.

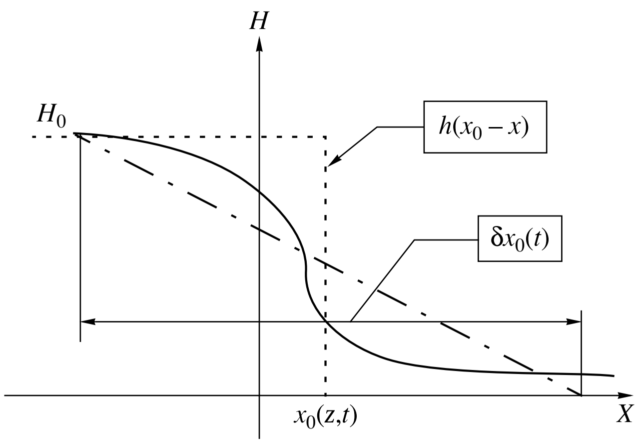

Fig. 4. A comparison of the position of the front of the propagating magnetic field (dashed lines) with the instantaneous magnetic field profile (solid curve). The linear approximation of the field front profile (dashed-and-dotted line) and the characteristic front width are shown schematically.

We differentiate (38) with respect to time, take into account (37), and perform simple manipulations to obtain

$$\frac{\partial x_0(z,t)}{\partial t} = D_\sigma \frac{\partial^2 x_0(z,t)}{\partial z^2} - \frac{cH_0}{8\pi e}\frac{\partial}{\partial z}\left(\frac{1}{n(z)}\right). \quad (39)$$

Equation (39) with the initial condition $x(z, t = 0) = 0$ has the solution

$$x_0(z,t) = -\frac{cH_0}{16\pi e}\int_{-\infty}^{\infty}\int_0^t \frac{\partial}{\partial \xi}\left(\frac{1}{n(\xi)}\right)\frac{e^{-\frac{(z-\xi)^2}{4D_\sigma(t-\tau)}}}{\sqrt{\pi D_\sigma(t-\tau)}}d\tau d\xi. \quad (40)$$

Taking the time derivative of (39), we find that the field front velocity, defined as $u \equiv u_x = \partial x_0/\partial t$, satisfies the equation

$$\frac{\partial u(z,t)}{\partial t} = D_\sigma \frac{\partial^2 u(z,t)}{\partial z^2}. \quad (41)$$

Assuming that, at the initial instant, the magnetic field does not diffuse in the z-direction $\left(\frac{\partial^2 H}{\partial x^2} \gg \frac{\partial^2 H}{\partial z^2}\right.$ and $\left.\left.\frac{\partial^2 x_0(z,t)}{\partial z^2}\right|_{t=0} = 0\right)$, we determine the front velocity at $t = 0$:

$$u(z,0) = -\frac{cH_0}{8\pi e}\frac{\partial}{\partial z}\left(\frac{1}{n(z)}\right). \quad (42)$$

The solution to equation (41) with the initial condition (42) is

$$u(z,t) = -\frac{cH_0}{16\pi e\sqrt{\pi D_\sigma t}}\int_{-\infty}^{\infty}\frac{\partial}{\partial \xi}\left(\frac{1}{n(\xi)}\right)e^{-\frac{(z-\xi)^2}{4D_\sigma t}}d\xi. \quad (43)$$

The same solution can also be derived by differentiating expression (40) with respect to time.

This solution generalizes the results obtained previously for the penetration depth of the magnetic field and the propagation velocity of the magnetic field front in plasmas with small [7] and infinitely large [9, 11, 13] density gradients. In fact, in the first case, we have

$$u = -\frac{cH_0}{16\pi e\sqrt{\pi D_\sigma t}}\int_{-\infty}^{\infty}\frac{\partial}{\partial \xi}\left(\frac{1}{n(\xi)}\right)e^{-\frac{(z-\xi)^2}{4D_\sigma t}}d\xi$$

$$= -\frac{cH_0}{8\pi e\sqrt{\pi D_\sigma t}}\int_{-\infty}^{+\infty}\frac{z-\xi}{4D_\sigma t}\frac{1}{n(\xi)}e^{-\frac{(z-\xi)^2}{4D_\sigma t}}d\xi.$$

Changing the variable $y = z - \xi$ and taking into account the relationship $\frac{1}{n(z-y)} \approx \frac{1}{n(z)} - \frac{1}{n^2(z)}\frac{\partial n}{\partial z}(z-y)$, which is valid for a plasma with a small density gradient, we arrive at expression (36). For a jump in the plasma density (Fig. 2), we have

$$\frac{\partial}{\partial \xi}\frac{1}{n} = -\frac{n_2-n_1}{n_1 n_2}\delta(\xi),$$

which yields

$$x_0(z,t) = -\frac{cH_0}{16\pi e D_\sigma}\frac{n_2-n_1}{n_1 n_2}$$
$$\times\left\{\sqrt{4D_\sigma t}e^{-\frac{z^2}{4D_\sigma}} - z\sqrt{\pi}\left[1-\operatorname{erf}\left(\frac{|z|}{\sqrt{4D_\sigma t}}\right)\right]\right\}, \quad (44)$$

where $\operatorname{erf}(x)$ is the error function. At the point $z = 0$, we obtain

$$x_0(t) = \frac{cH_0}{16\pi e D_\sigma}\frac{n_2-n_1}{n_1 n_2}\sqrt{D_\sigma t} = \frac{\sigma H_0}{2ec}\frac{n_2-n_1}{n_1 n_2}\sqrt{D_\sigma t}, \quad (45)$$

thereby determining the effective diffusion coefficient for the magnetic field,

$$D_{\text{eff}} = \left(\frac{\sigma H_0}{2ec}\frac{n_2-n_1}{n_1 n_2}\right)^2 D_\sigma, \quad (46)$$

which was evaluated earlier by Vikhrev and Zabajdullin [11, 13].

If the plasma density increases gradually from n_1 to n_2 over a finite distance, then, as $t \longrightarrow \infty$ ($4/D_\sigma t \gg \xi^2$),

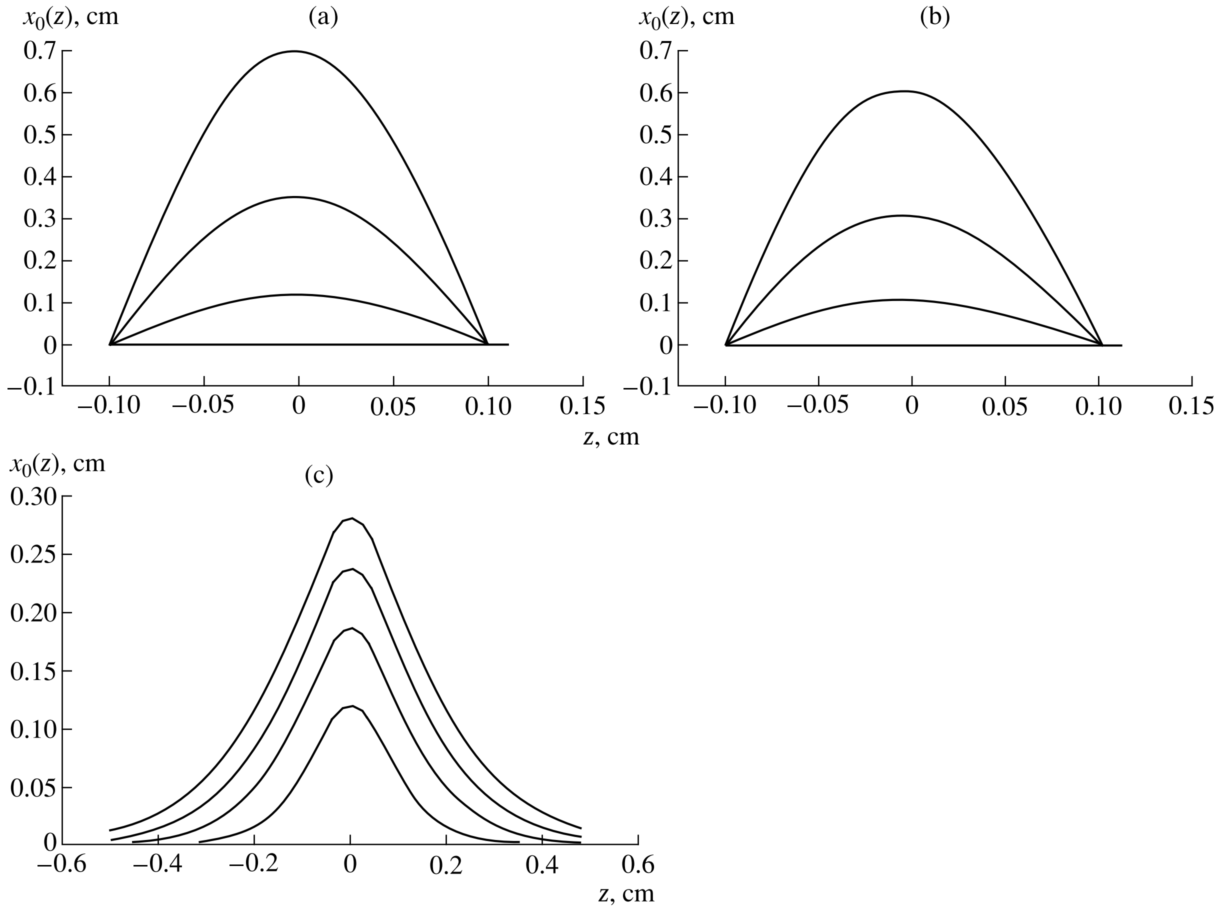

Fig. 5. A comparison between the spatial profiles of the magnetic field front at different times $\left(t = \frac{n}{4} \times 10^{-7}\text{ s}, n = 1, \ldots, 4\right)$. Profiles (a) and (b) are computed from the approximate formulas (24) and (25), respectively, and profile (c) is calculated from the exact formula (40). The plasma density profile used in simulations is specified as $n(z) = n_0(1 + 1/2\delta(1 - \sin(\pi z/L)))$ for $|z| < L/2$ and $n = 0$ for $|z| \geq L/2$. Here, $n_0 = 1.0 \times 10^{17}$ cm^{-3}, $L = 0.2$ cm, the relative jump in the density is $\delta = 0.1$, the electron temperature is $T_e = 5$ eV, the initial magnetic field is $H_0 = 1.0 \times 10^4$ G, $D_\sigma = 3.7 \times 10^5$ cm^2/s, and $\omega_e\tau_{ei} = 33.5$.

the front velocity will approach the value

$$u = -\frac{cH_0}{16\pi e\sqrt{\pi D_\sigma t}}\frac{n_2 - n_1}{n_2 n_1}. \tag{47}$$

Numerical integration of expression (40) shows that the approximate formulas (24) and (25) are insufficiently accurate, because the magnetic field diffusion in the direction perpendicular to the instantaneous magnetic field front is incompletely incorporated (Fig. 5).

3.2. Front Width

In order to describe the front width δx_0, it is expedient to define it as

$$H_0(\delta x_0(z, t))^2 = \alpha\int_{-\infty}^{\infty}(x - x_0)[H(x, z, t) - H_0 h(x_0 - x)]dx. \tag{48}$$

It is convenient to choose the proportionality coefficient α by replacing the function $H(x, z, t)$ in formula (48) with its linear approximation (Fig. 4). As a result, we approximately obtain $\alpha = 6$.

As an example, we can use definition (48) taken with the exact field magnitude (34) corresponding to a small density gradient as an adequate characteristic front width, in which case we have $\delta x_0 = \pi\lambda$.

To transform (48), we apply the same mathematical procedure as in the previous section. As a result, we arrive at the differential equation for the front width,

$$\begin{aligned}\frac{1}{\alpha}\frac{\partial(\delta x_0)^2}{\partial t} &= \frac{1}{\alpha}\frac{\partial^2(\delta x_0)^2}{\partial z^2} + D_\sigma\left[1 + \left(\frac{\partial x_0}{\partial z}\right)^2\right] \\ &+ \frac{c}{4\pi e n H_0}\frac{\partial \ln n}{\partial z}\int_{-\infty}^{\infty}(x - x_0)H\frac{\partial H(x, z, t)}{\partial x}dx,\end{aligned} \tag{49}$$

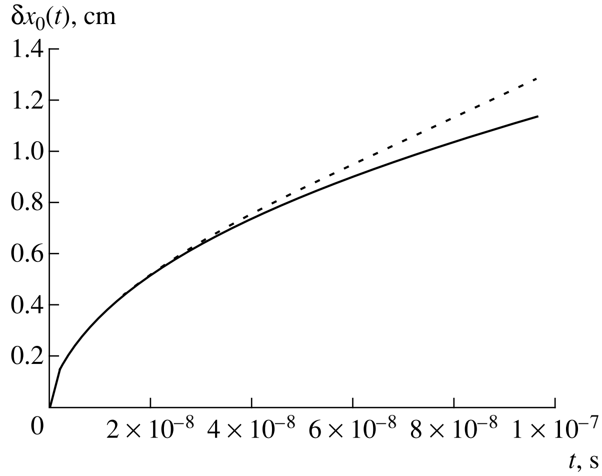

Fig. 6. Time evolution of the front width at the point $z = 0$ in the case of a periodically varying density profile $n(z) = n_0(1 + 1/2\delta(1 - \sin(\pi z/L)))$. The remaining parameters, namely, n_0, L, δ, T_e, and H_0, are the same as in Fig. 5, the initial front width being $q = 0$. The dashed curve reflects the time evolution computed from the approximate formula (54).

with the initial condition

$$H_0(\delta x_0)^2 = \alpha \int_{-\infty}^{\infty} x[H(x, z, t = 0) - H_0 h(-x)]dx. \quad (50)$$

The derivative $\partial x_0/\partial z$ can be found from (40).

Approximating the magnetic field profile in the last term of equation (49) by a linear function, we can readily see that, under the conditions $(L/\delta x_0)\omega_e\tau_{ei} \gg 1$ (where L is the spatial scale on which the plasma density varies) and $\omega_e\tau_{ei} \gg 1$ (which is characteristic of the enhanced-rate propagation problem under discussion), this term may be neglected. Interestingly, with the exact solution (34), the last term in equation (49) exactly equals zero.

Inserting expression (40) for $x_0(z, t)$ into (49) yields the equation

$$\frac{1}{\alpha}\frac{\partial(\delta x_0)^2}{\partial t} \simeq D_\sigma\frac{1}{\alpha}\frac{\partial^2(\delta x_0)^2}{\partial z^2} + D_\sigma\left[1 + \frac{1}{4}(\omega_e\tau_{ei}D_\sigma)^2\left(\int_{-\infty}^{\infty}\int_0^t \frac{\partial^2}{\partial\xi^2}\left(\frac{n_0}{n(\xi)}\right)\frac{e^{-\frac{(z-\xi)^2}{4D_\sigma t}}}{\sqrt{4\pi D_\sigma\tau}}d\tau d\xi\right)^2\right]. \quad (51)$$

With allowance for (50), the solution to this equation can be written as

$$(\delta x_0(z, t))^2/\alpha = D_\sigma t + \frac{1}{\sqrt{4\pi D_\sigma t}}\int_{-\infty}^{+\infty} q(\xi)e^{-\frac{(z-\xi)^2}{4D_\sigma t}}d\xi + D_\sigma\int_0^t\int_{-\infty}^{+\infty} g^2(\xi, \tau)\frac{e^{-\frac{(z-\xi)^2}{4D_\sigma(t-\tau)}}}{\sqrt{4\pi D_\sigma(t-\tau)}}d\xi d\tau, \quad (52)$$

where

$$g(z, t) \equiv \frac{\partial x_0}{\partial z} = \frac{1}{2}\omega_e\tau_{ei}D_\sigma \times \int_{-\infty}^{\infty}\int_0^t \frac{\partial^2}{\partial\xi^2}\left(\frac{n_0}{n(\xi)}\right)\frac{e^{-\frac{(z-\xi)^2}{4D_\sigma t}}}{\sqrt{4\pi D_\sigma t}}d\tau d\xi \quad (53)$$

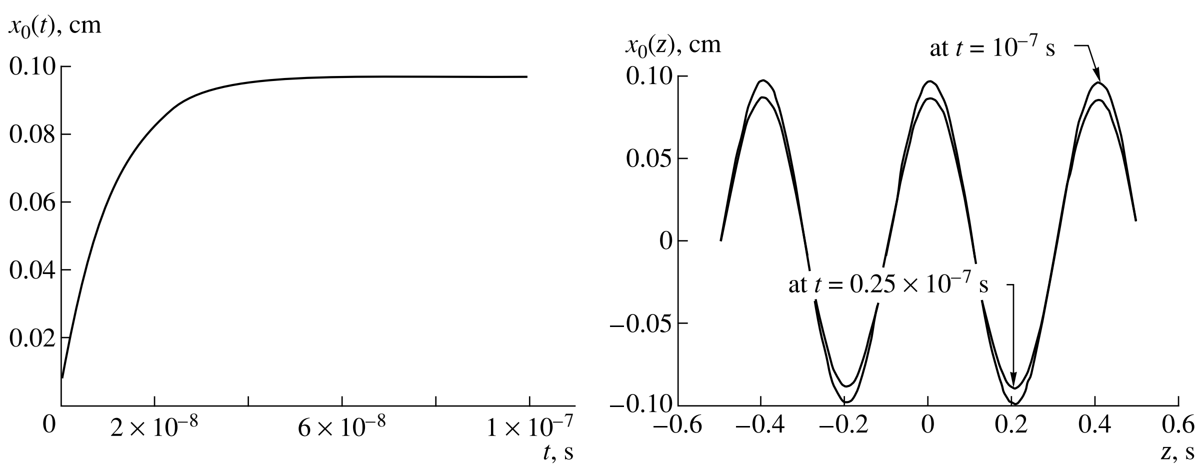

Fig. 7. Time evolution of the effective magnetic field front (40) at the fixed point $z = 0$ in the case of a plasma with a periodically varying density (on the left) and spatial profiles of the field front at two different times (on the right) for the same parameters as in Fig. 6.

and the quantity $q(z) \equiv (\delta x_0(z))^2/\alpha|_{t=0}$ accounts for the contribution of the initial thickness of the magnetic field front.

If the second derivative $\partial^2/\partial z^2[n_0/n(z)]$ varies more gradually than $\exp\left(-\frac{z^2}{4D_\sigma t}\right)$ (at least over the distance between $z - \sqrt{12D_\sigma t}$ and $z + \sqrt{12D_\sigma t}$), then, in expressions (52) and (53), we can take the density-dependent functions outside the integrals. For q = const, we obtain from (52) and (53)

$$\delta x_0(z, t) = \alpha\sqrt{\frac{q}{\alpha} + \left\{1 + \left[\frac{1}{6}(\omega_e\tau_{ei})D_\sigma t\frac{\partial^2}{\partial z^2}\left(\frac{n_0}{n(z)}\right)\right]^2\right\}D_\sigma t}. \quad (54)$$

The condition for the function $\partial^2/\partial z^2[n_0/n(z)]$ to gradually vary yields

$$\left|\frac{\partial^2}{\partial z^2}\left(\frac{n_0}{n(z \pm \sqrt{12D_\sigma t})}\right) - \frac{\partial^2}{\partial z^2}\left(\frac{n_0}{n(z)}\right)\right| \ll 1. \quad (55)$$

This inequality determines the time interval over which expression (54) is valid.

Numerical simulations with formula (52) and its approximate version (54) show that, in the initial stage, the approximate expression (54) gives quite exact (up to three significant digits) results (see Fig. 6).

It is of interest to consider the case of a plasma whose density varies periodically in space. Our simulations carried out with formula (40) show that, in such a plasma, the shape of the field front profile changes markedly only over a finite time interval and then remains essentially unchanged (Fig. 7). Of course, this does not indicate that the magnetic field stops penetrating: the field front becomes thicker by an amount δx_0, which is determined from (52), (53), or (54).

4. CONCLUSION

We have developed two qualitatively different EMHD models of an enhanced-rate (in comparison with ordinary diffusion) propagation of a magnetic field in a plasma due to the Hall effect. The first model is based on a simple hydrodynamic approach, which, in our opinion, has permitted considerable insights into the role of the Hall effect in a plasma. In particular, this model makes it possible to reproduce some familiar theoretical results and may prove useful for clarifying the role of the Hall effect without turning to simplified models, which are inevitably used in rigorous analytic analyses.

In contrast, the second model endeavors to provide an exact analytic description of the representative parameters of the magnetic field propagation. In the case of an enhanced-rate propagation of the magnetic field in an isothermal inhomogeneous plasma, these are the effective velocity of the magnetic field front and the effective front width. The results obtained with this model make it possible to check the accuracy of the simple formulas—in particular, formula (25), derived by Zabajdullin [22], and formula (24), obtained in Section 2—that describe a transition from the regime of diffusive penetration [9] to the regime in which the magnetic field propagates as a wave [7].

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